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HOPF BIFURCATION SUBJECT TO A LARGE DELAY

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Abstract

Hopf bifurcation theory for an oscillator subject to a weak feedback but a large delay is investigated in detail. The problem is motivated by semiconductor laser instabilities which are generated by optical feedbacks. A specific laser problem is examined for its Hopf bifurcation. Because of the large delay, a delayed amplitude appears in the slow time amplitude equation and leads to new bifurcations to periodic and quasiperiodic states. We determine expressions for all bifurcation points and show how the bifurcation points to quasiperiodic oscillations emerge from double Hopf bifurcation points. Different cases of bistability between steady, periodic and quasiperiodic regimes are possible which we study numerically. Finally, the validity of the Hopf bifurcation approximation is investigated numerically by comparing the bifurcation diagrams of the original laser equations and the slow time amplitude equation.

Keywords: semiconductor laser instabilities, system of delay-differential equations, two-time solution, bifurcation to periodic and quasiperiodic oscillations.

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1   Introduction

Semiconductor lasers are the technology of choice for many applications requiring a coherent light source because they are of relatively small size, they are massively produced and they are easy to operate. Applications of semiconductor lasers span a broad range of areas from optical communication to optical ranging and sensing. However, semiconductor lasers are extremely sensitive to optical feedback (OFB). Small amount of OFB may result from the reflection from an optical disk or from the end of an optical fiber. This feedback perturbs the normal output of the laser and generates dynamical instabilities. These instabilities are typically accompanied by unwanted higher intensity of frequency noise. It is therefore of practical interest to understand how the stability of the laser depends on OFB. A simple model has been formulated in 1980 by Lang and Kobayashi (LK) and consists of two nonlinear ordinary delay-differential equations for the complex electrical field $Y(t)$ and the carrier number $Z(t)$ [?]. Time t is measured in units of the photon lifetime τ_p. A term proportional to $Y(t-\tau)$ appears in the field equation and describes the effect of OFB after the reflection of light from a mirror. τ is proportional to the round-trip time from laser to mirror and back to laser. Computer simulations have shown that the LK delay-differential equations correctly describe the dominant effects observed experimentally. This includes the occurrence of mode hopping [?, ?], coexisting time-periodic attractors [?, ?, ?] and different forms of chaotic attractors called low-frequency fluctuations [?, ?, ?]. More recently, a numerical bifurcation study suggests that LK instabilities may result from cascading bifurcations starting from a Hopf bifurcation [?]. Ideally, LK equations models an experimental set-up consisting of a laser exposed to weak reflection of light from a mirror located at a sufficiently large distance to avoid multiple reflections (one to two meters). This implies a round-trip time τ' from laser to mirror and back to laser of the order of 1 ns which is relatively large compared to the photon lifetime $\tau_p \sim 1$ ps (i.e., a dimensionless delay $\tau \equiv \tau'/\tau_p = O(10^3)$). It is this large delay τ which is causing the laser instabilities even for low feedback rates and which motivates our mathematical interest for the Hopf bifurcation subject to a large delay.

Periodic solutions of delay-differential equations and their behavior for

large delay have been studied mathematically for equations of the form

$$u' = f(u, u(t - \tau), \lambda) \tag{1}$$

where $u \in R$. Applications appear in the biological sciences [?], chemical problems [?] and in nonlinear optics [?, ?]. Most of the solutions computed numerically correspond to square-wave oscillations exhibiting a period close to τ as $\tau \to \infty$. The condition for the bifurcation to these square-wave periodic regimes has been examined mathematically in [?, ?]. By contrast to these square-wave oscillations, the oscillatory solution of LK equations for low feedback is nearly harmonic in time [?, ?] and the frequency of the oscillations is $O(1)$ as $\tau \to \infty$. The effect of a large τ appears through a secondary bifurcation to quasiperiodic oscillations which exhibits a low frequency proportional to τ^{-1}. This typical two-time behavior of the solution of LK problem for low feedback rates but large delays motivates the study of specific Hopf bifurcation problems of the form

$$u' = f(u, \epsilon^2 u(t - \tau), \lambda) \tag{2}$$

where $u \in R^n$ ($n \geq 2$). In Eq. (??), ϵ and τ represent a small parameter and a large parameter, respectively. General Hopf bifurcation theories for systems of delay-differential equations are difficult (see [?]) and we shall take advantage of the small parameter ϵ. Specifically, we assume that the $\epsilon = 0$ problem admits a Hopf bifurcation at $\lambda = \lambda_0$ from $u = 0$ to a stable time-periodic state. Our objective is to investigate this bifurcation for small ϵ and progressively larger values of τ. To this end, we may apply a two-time perturbation method introducing a fast time t and a slow time $\nu \equiv \epsilon^2 t$ [?, ?]. The two independent time variables imply that $u(t) = u(t, \nu)$ and $u(t - \tau) = u(t - \tau, \nu - \epsilon^2 \tau)$. If τ is $O(1)$, $u(t - \tau, \nu - \epsilon^2 \tau)$ simplifies as

$$u(t - \tau, \nu - \epsilon^2 \tau) = u(t - \tau, \nu) + O(\epsilon^2) \tag{3}$$

and the effect of the delay on the slow time ν will not appear in the leading order amplitude equation. On the other hand, if τ is $O(\epsilon^{-2})$ large, we cannot expand $u(t - \tau, \nu - \epsilon^2 \tau)$ for small ϵ and the slow time delay given by $\epsilon^2 \tau$ will appear explicitly in the bifurcation equation through a delayed amplitude. As we shall demonstrate, this slow time delay is responsible for multiple branching of solutions.

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In this paper, we concentrate on a simple laser problem which is modeled by equations of the form of (??). This will allow us to discuss the validity of Hopf local theory by comparing the numerical bifurcation diagrams obtained from the original equations and from the slow-time amplitude equation. Specifically, we shall consider a semiconductor laser subject to a delayed feedback controlled opto-electronically [?]. The problem is mathematically simpler than LK problem because the phase of the laser field does not play a role and the dynamical variables reduce to the intensity of the laser field and the electronic carrier density. The laser equations are formulated in dimensionless form in Section 2 and the slow-time amplitude equation is derived in Section 3. In Section 4, we determine its steady state solutions and formulate stability conditions which we investigate numerically. We conclude that multiple branches of periodic and quasiperiodic states are possible and are connected at double Hopf bifurcation points. In Section 5, we evaluate the validity of our Hopf bifurcation analysis by comparing numerical bifurcation diagrams. We also identify different forms of bistability between steady, periodic and quasiperiodic regimes which are of particular interest for experiments. Finally, Section 6 discusses the physical relevance of our results for semiconductor lasers instabilities.

2 Formulation

The semiconductor laser system and its optoelectronic feedback is sketched in Figure 1. Part of the laser output is detected with a high-speed photodetector. The detector photocurrent component is selected with a T bias, amplified and added to the DC pump current [?]. In dimensionless form, the laser rate equations modeling this system are given by [?]

$$\frac{dY}{dt} = (1 + i\alpha)ZY, \tag{4}$$

$$T\frac{dZ}{dt} = P + \eta\left|Y(t-\tau)\right|^2 - Z - (1 + 2Z)\left|Y\right|^2 \tag{5}$$

where Y and Z represent the complex electrical field and the electronic carrier density. Time t is measured in units of the photon lifetime τ_p ($t \equiv t'/\tau_p$). α is defined as the linewidth enhancement factor, $T \equiv \tau_n/\tau_p$ is a ratio of two time constants where τ_n is defined as the carrier lifetime, $\tau \equiv \tau'/\tau_p$ is the

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dimensionless delay of the feedback where τ' is the cavity transit time and P is the pump parameter above threshold . These equations are the traditional semiconductor laser rate equations with P now replaced by $P + \eta |Y(t - \tau)|^2$ which models the effect of the DC coupled optoelectronic feedback. Typical values of the fixed parameters are $\alpha \sim 3 - 6$, $\tau_p \sim 2 \ ps$, $\tau_n \sim 2 \ ns$ and $\tau \sim 1 - 10 \ ns$ which mean large, $O(10^3)$ values, for both T and τ.

We shall take advantage of these two large parameters in order to determine an approximation of the solution of Eq. (??) and (??). Introducing the decomposition $Y = \sqrt{I} \exp(i\psi)$ allow us to reduce Eqs. (??) and (??) to two coupled equations for the intensity I and the carrier density Z. ψ is passively related to I and Z. Furthermore, the large T parameter which multiplies the left hand side of the carrier equation (??) can be removed by a change of variables. Introducing the new variables x, y and s defined by

$$I = P(1 + y), \ Z = \frac{\omega}{2}x, \ s = \omega t, \tag{6}$$

where

$$\omega \equiv \sqrt{2P/T} \tag{7}$$

is known as the laser relaxation oscillation frequency, the two equations for I and Z become

$$x' = -y - \xi x \left(1 + \frac{2P}{1 + 2P}y\right) + \eta \left[1 + y(s - \theta)\right], \tag{8}$$

$$y' = (1 + y)x \tag{9}$$

where prime means differentiation with respect to s. The new variables x and y represent the deviations of I and Z from their steady state values $I = P$ and $Z = 0$, respectively. The new parameters ξ and θ are defined by

$$\xi \equiv \omega \frac{(1 + 2P)}{2P} \text{ and } \theta \equiv \omega\tau. \tag{10}$$

ξ is known as the damping rate of the $2\pi/\omega$ laser intensity oscillations and θ is a rescaled delay. The large T parameter now appears in the small parameter ξ (ξ is proportional to ω which is $O(T^{-1/2})$ small). Eqs. (??) and (??) are in an appropriate form for an asymptotic analysis. These equations are equivalent to the equations studied in [?, ?] for a variety of different optoelectronic devices. In [?, ?], the objective was to determine *strongly pulsating*

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periodic solutions for *moderate values* of the delay. Here, we determine periodic solutions which are nearly *harmonic* for *large values* of the delay. Recent attempts to investigate the large delay limit for similar laser equations are given in [?, ?]. Simplified amplitude equations were derived but their numerical validity as providing approximations of the solutions of the original laser equations have not been examined. In this paper, we determine analytical approximations of several branches of periodic solutions and their validity is investigated in detail.

3   Hopf bifurcation

In this section, we determine a slow time amplitude equation by a two-time analysis of Eqs. (??) and (??) based on the limit ξ and η small but θ large. The perturbation analysis is quite simple and we summarize the main results. We first introduce a small parameter ϵ defined by

$$\epsilon \equiv \sqrt{\xi} \tag{11}$$

and assume the following scalings for the parameters η and θ

$$\eta(\epsilon) = \epsilon^2 C, \ \theta = \epsilon^{-2}\Theta. \tag{12}$$

We seek a two-time solution of the form

$$x(s, \nu, \epsilon) = \epsilon x_1(s, \nu) + \epsilon^2 x_2(s, \nu) + \dots \tag{13}$$

$$y(s, \nu, \epsilon) = \epsilon y_1(s, \nu) + \epsilon^2 y_2(s, \nu) + \dots \tag{14}$$

where ν is a slow time variable defined by

$$\nu \equiv \epsilon^2 s. \tag{15}$$

Inserting (??)-(??) into Eqs. (??) and (??), using the chain rule $d/ds = \partial/\partial s + \epsilon^2 \partial/\partial \nu$ and noting that $y(s - \theta) = y(s - \theta, \nu - \Theta)$, lead to a sequence of problems for the unknown coefficients in (??) and (??). The solution of each problem is easily determined. We find that

$$\begin{aligned} x &= \epsilon\left(A(\nu)\exp(is) + c.c.\right) + O(\epsilon^2), \\ y &= -\epsilon\left(iA(\nu)\exp(is) + c.c.\right) + O(\epsilon^2) \end{aligned} \tag{16}$$

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where *c.c.* means complex conjugate and the slow-time amplitude A satisfies the following ordinary differential equation

$$A' = \frac{1}{2}\left[iCA - \frac{i}{3}A^2 A^* - A - iCA(\nu - \Theta)\exp(-i\theta)\right]. \qquad (17)$$

The main difference with the case $\theta = O(1)$ is the presence of the delayed amplitude $A(\nu - \Theta)$ in Eq. (??) instead of $A(\nu)$. It is instructive to review the significance of each term in the right hand side of Eq. (??). The first two terms in the right hand side of Eq. (??) represent the linear and nonlinear correction to the frequency and could be anticipated from a study of the laser equations (??) and (??) without damping ($\xi = 0$) and without feedback ($\eta = 0$). The third term in the right hand side of Eq. (??) comes from the coefficient multiplying ξ in Eq. (??). It represents the physical damping of the laser oscillations. Finally the last term in the right hand side of Eq. (??) is the contribution of the feedback.

Eq. (??) is the starting point of our bifurcation analysis. For clarity, the different parameters introduced in the last two sections are shown in Table 1.

Symbol	Definition	Value	
ω	$\sqrt{\frac{2P}{T}}$	0.03	relax. oscill. frequency
$\epsilon^2 = \xi$	$\omega\frac{1+2P}{2P} = \frac{1+2P}{\sqrt{2PT}}$	0.06	damping rate
θ	$\omega\tau = \sqrt{\frac{2P}{T}}\tau$	31.62 to 63.25	scaled delay
Θ	$\epsilon^2\theta = \frac{1+2P}{T}\tau$	2 to 4	scaled delay
C	η/ϵ^2	-1.6 to 1.6	scaled feedback rate

Table 1. Laser paramerers and their typical values. $T = 1000$, $P = 0.5$ and $\tau = 1000$ to 2000.

4 Analysis of the slow time amplitude equation

In this section, we determine the steady state solutions of Eq. (??) and investigate their linear stability properties. The validity of Eq. (??) as an approximation of the solutions of the original laser equations (??) and (??) will be investigated numerically in Section 5. The main advantage of Eq. (??) is the fact that periodic and quasiperiodic solutions of the original laser

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equations now correspond to steady and periodic solutions for the amplitude $|A|$. It is mathematically convenient to introduce the decomposition $A = R \exp(i\phi)$ into Eq. (??). We then obtain the following equations for R and ϕ

$$R' = \frac{1}{2} \left[-R + CR(\nu - \Theta) \sin\left(-\theta + \phi(\nu - \Theta) - \phi \right) \right], \tag{18}$$

$$R\phi' = \frac{1}{2} \left[CR - CR(\nu - \Theta) \cos\left(-\theta + \phi(\nu - \Theta) - \phi \right) - \frac{1}{3} R^3 \right]. \tag{19}$$

Before we analyze these equations in detail, it is interesting to determine the solution of the regular problem with $\theta = O(1)$. Equivalently, we solve Eqs. (??) and (??) with $\Theta = 0$. A periodic solution of the original laser equations corresponds to $R = cst$ and $\phi = \phi(0) + B\nu$. From Eqs. (??) and (??) with $\Theta = 0$, we find that

$$C = C_H^0 \equiv -\frac{1}{\sin(\theta)} \tag{20}$$

and

$$R = \sqrt{6(B_0 - B)} \tag{21}$$

where B_0 is defined by

$$B_0 \equiv -\frac{1}{2} \tan(\frac{\theta}{2}) \tag{22}$$

and $B < B_0$ is arbitrary. $C = C_H^0$ is a Hopf bifurcation point and the bifurcation is vertical at this order of the analysis. A higher order analysis will lead to the direction of bifurcation but we do not need this result (see Appendix in [?]). Our main observation is that the Hopf bifurcation is unique if the delay θ is an $O(1)$ quantity. As we shall now demonstrate, increasing θ will bend the vertical bifurcation branch (??)-(??) and produce new (bifurcation and isolated) branches of periodic solutions.

4.1 Periodic solutions

We consider Eqs. (??) and (??) with $\Theta \neq 0$ and seek a solution of the form $R = cst$ and $\phi = \phi(0) + B\nu$. This solution corresponds to a periodic state of the original laser equations which exhibits a frequency equals to $1 + \epsilon^2 B$. The exact solution for $R^2 = R^2(C)$ and $B = B(C)$ can be formulated in parametric form as

$$C = -\frac{1}{\sin\left(\theta + B\Theta \right)}, \tag{23}$$

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$$R^2 = 3\left[-2B - \tan\left((\theta + B\Theta)/2\right)\right] > 0 \tag{24}$$

where the frequency B is the parameter. Changing B from $-\infty$ to $+\infty$, we obtain C and R using (**??**) and (**??**). See Figure 2. The figure show that there exist several branches of solutions. The limit as $\Theta \to 0$ is discussed in Appendix A.

From (**??**) and (**??**) with $R = 0$, we find that the Hopf bifurcation points $(B, C) = (B_H, C_H)$ satisfy Eq. (**??**) and

$$2B + \tan\left((\theta + B\Theta)/2\right) = 0. \tag{25}$$

Thus, we determine $B = B_H$ by solving Eq. (**??**) and then obtain $C = C_H$ using (**??**). A different and more physically interesting way to investigate the roots of Eq. (**??**) as a function of θ is to study the implicit expression given by

$$\theta = \frac{2}{1 + B\xi}\left[m\pi - \arctan(2B)\right] \tag{26}$$

where $m = 0, 1, 2...$ Having $\theta = \theta(B)$, we determine $C = C_H$ by (**??**). Thus, we obtain θ and C_H by changing parameter B continuously from $-\infty$ to ∞. See Figure 3. For each m , we have a line of bifurcation points which corresponds to a different branch of periodic solutions.

The direction of bifurcation can be determined from an analysis of Eqs. (**??**) and (**??**) for (B, C) close to (B_H, C_H). We find that the bifurcation is defined for $C > C_H$ if

$$\cos(\theta + B_H\Theta) < 0, \tag{27}$$

or equivalently, for $C < C_H$ if $\cos(\theta + B_H\Theta) > 0$. The direction of bifurcation changes if $\cos(\theta + B_H\Theta) = 0$ which implies, using (**??**), that $C_H = 1$. Since $C_H = 1$ at the minimum of the curve $C_H = C_H(\theta)$, the change of direction occurs exactly at that point. The successive minima are located at $\theta = \theta_m$. If $C_H > 0$, we have verified that the bifurcation is supercritical $(C > C_H)$ if $\theta < \theta_m$ and is subcritical $(C < C_H)$ if $\theta > \theta_m$.

Figure 3a also shows that distinct Hopf bifurcation lines may intersect. These points correspond to degenerate Hopf bifurcation points which exhibit two distinct periodic modes (double Hopf bifurcation points). From these points, the solution is in first approximation a linear combination of the two periodic modes and multiple branching of solutions are possible at these points. We anticipate that in addition to the two pure mode bifurcations, mixed mode bifurcations are possible. We verify this numerically and

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analytically in the next subsection. This mixed mode regime is typically quasiperiodic exhibiting the two pure mode frequencies.

By contrast to the Hopf points, the limit points $(B, C) = (B_L, C_L)$ lead to isolated branches of periodic solutions. They satisfy the condition $\sin(\theta + B_L \Theta) = \pm 1$ and they are located at

$$C_L = \pm 1 \text{ and } B_L = \frac{1}{\Theta}\left(\mp(1 + 4n)\pi/2 - \theta\right) \tag{28}$$

where $n = 0, 1, 2, ...$

4.2 Quasiperiodic bifurcations

We next investigate the stability of the periodic states given by (**??**) and (**??**). A detailed stability analysis of these solutions is difficult because the characteristic equation is transcendental. But we may examine the case of small eigenvalues and the case of purely imaginary eigenvalues.

From the linearized equations, we determine the characteristic equation for the growth rate σ. Assuming σ small, we then obtain an expression for σ of the form

$$\sigma \simeq \alpha \cos(\theta + B\Theta)R^2 \tag{29}$$

where α is always positive. If all the remaining eigenvalues have a negative real part, (**??**) implies that the Hopf bifurcation branch is stable near its bifurcation point $C = C_H$ (limit $R \to 0$, $\cos(\theta + B\Theta)$ fixed) provided that (**??**) is satisfied, i.e., for a supercritical Hopf bifurcation. The expression (**??**) also implies the local stability of one of the two branches of periodic solutions that emerges from a limit point $C = C_L$ (recall that $\cos(\theta + B\Theta) = 0$ at $C = C_L$ and consider the limit $|\cos(\theta + B\Theta)| \to 0$, R fixed of Eq. (**??**)).

We next examine the conditions for a Hopf bifurcation from the $R = cst$ solution meaning a secondary bifurcation to quasiperiodic oscillations for the laser intensity. Substituting $\sigma = i\mu$ into the characteristic equation, we obtain two conditions for the critical feedback rate C and frequency μ. They are given by

$$\begin{aligned}
-4\mu^2 + C^2\left((\cos(\mu\Theta) - 1)^2 - \sin^2(\mu\Theta)\right) \\
+ (\cos(\mu\Theta) - 1)C\cos(\theta + B\Theta)\tfrac{2}{3}R^2 \\
- \sin(\mu\Theta)4\mu = 0
\end{aligned} \tag{30}$$

and

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$$-2C^2(\cos(\mu\Theta) - 1)\sin(\mu\Theta)$$
$$-4\mu\left(\cos(\mu\Theta) - 1\right)$$
$$-\sin(\mu\Theta)C\cos(\theta + B\Theta)\tfrac{2}{3}R^2 = 0 \tag{31}$$

where R and B are related to C by (??) and (??). The solution of these equations has been determined numerically. See Figure 4. We find successive lines of quasiperiodic bifurcation points that all emerge from the double Hopf bifurcation points. These points are denoted by P_1 and P_4 in Figure 4. In Appendix B, we show analytically that these lines emerge from the double Hopf bifurcation points with the new frequency $\mu = B_2 - B_1$. B_1 and B_2 denote the two Hopf frequencies at the double Hopf point. This is consistent with a bifurcation theory near a multiple eigenvalue since we expect a solution of the form

$$A \sim R_1 \exp\left(i(B_1\nu + \phi_1)\right) + R_2 \exp\left(i(B_2\nu + \phi_2)\right) \tag{32}$$

near this point. At the end of Appendix B, we show that the secondary bifurcating solution has indeed the form of (??). A detailed analysis of the solutions near the double Hopf bifurcation point is beyond the scope of this paper devoted to a single Hopf bifurcation subject to a large delay. In the next section, we investigate numerically the bifurcation possibilities suggested by Figure 4 and emphasizes cases of coexisting stable states (bistability).

5   Numerical Bifurcation diagrams.

The objectives of this section are twofold. First, we investigate numerically several cases of coexisting (steady or not) attractors. These possibilities are of physical interest because they can be observed experimentally. Second, we examine the validity of our approximations by comparing the numerical bifurcation diagram of the original laser equations and the bifurcation diagram obtained from the amplitude equation.

We first determine the bifurcation diagram of the solutions of Eq. (??) for different values of θ. The main advantage of this equation is that steady and periodic states correspond to periodic and quasiperiodic states of the original laser equations. The simplicity of Eq. (??) allowed us to have analytical expressions for all the steady state branches. We shall limit our analysis to the range $60 < \theta < 70$ and consider $C > 0$. We follow the predictions of Figure

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4 and our results are summarized by the three qualitative diagrams shown in Figure 5 and by the detailed diagram given in Figure 6. In Figure 5, we concentrate on the gradual change of the first Hopf bifurcation branch as the delay θ is progressively increased. Figure 6 examines a case of two coexisting branches of periodic solutions. All our diagrams are based on the numerical determination of the stable solutions of the slow time amplitude equations (**??**) and (**??**) as well as the exact analytical expressions of the branches of periodic solutions given by (**??**) and (**??**). In Figure 6, we determine the stable solutions from the amplitude equations as well as from the original laser equations. Four key points in Figure 4 mark qualitative changes of the bifurcation diagrams. As described in Section 4.2, P_1 and P_4 are double Hopf bifurcation points where lines of secondary bifurcation points appear. P_2 refers to the change of direction of the Hopf bifurcation and was analyzed in Section 4.1. P_3 verifies the condition $C_H = C_{QP}$ but corresponds to distinct bifurcation points (different amplitudes of the solution).

The simplest case occurs if $\theta_{P_1} < \theta < \theta_{P_2}$ (Figure 5a). A supercritical Hopf bifurcation is followed by a secondary bifurcation to quasiperiodic oscillations. If $\theta_{P_2} < \theta < \theta_{P_3}$ (Figure 5b), the sequence of bifurcations is similar to the sequence shown in Figure 5a except that the Hopf bifurcation is subcritical and allows the coexistence of a stable steady state and a stable periodic regime. This coexistence becomes richer if $\theta_{P_3} < \theta < \theta_{P_4}$ (Figure 5c): in addition to coexisting steady and periodic states, we may now have coexisting stable steady and quasiperiodic regimes. If $\theta > \theta_{P_4}$, new branches of periodic and quasiperiodic solutions appear. An example is studied quantitatively in Figure 6.

Specifically, we compare the bifurcation diagrams of the periodic and quasiperiodic states obtained from the full laser equations (**??**) and (**??**) (Figure 6a) and from the slow time amplitude equations (**??**) and (**??**) (Figure 6b). In Figure 6b, we observe a stable Hopf bifurcation branch that emerges from zero at $\eta \sim 0.076$. This bifurcation is followed by a secondary bifurcation to a branch of stable quasiperiodic oscillations at $\eta \sim 0.086$. Near $\eta \sim 0.1$, we note a tertiary bifurcation to a more complicated state. In addition to the Hopf bifurcation branch, there exists an isolated branch of periodic solutions that appears from a limit point located near $\eta \sim 0.064$. This branch then exhibits a secondary bifurcation to a quasiperiodic state at $\eta \sim 0.068$ which is itself followed by a tertiary bifurcation to more complex quasiperiodic oscillations ($\eta \sim 0.082$). We compare this bifurcation diagram to the diagram

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obtained by solving directly the laser equations (Figure 6a). The agreement is excellent for the low amplitude branches. The agreement becomes more qualitative for the higher amplitude branches of periodic solutions, as we may expect, but the same sequence of bifurcation transitions is observed in both diagrams. Note that $\epsilon = 0.17$ for our parameter values. We have verified that the numerical agreement between diagrams improves for lower ϵ (higher T).

6   Discussion

The main objective of this paper was to evaluate the validity of Hopf bifurcation theory for a system of two delay-differential equations modeling semiconductor lasers subject to a weak optical feedback but exhibiting a large delay. To this end, we have compared the numerical bifurcation diagrams obtained from the original laser equations and from the slow time amplitude equations. In addition, we have determined useful analytical expressions for branches of periodic solutions and formulate conditions for their bifurcations. These expressions considerably helped the numerical study. Previous attempts [?, ?] have tried to take advantage of this large delay in order to derive simplified laser equations but have never estimated the asymptotic validity of their approximations.

An important result of our bifurcation analysis is the observation that a large delay may generate multiple branches of periodic and quasiperiodic oscillations even for low feedback. These multiple Hopf or isolated branching of solutions are predicted by our amplitude equation. Because the derivation of this amplitude equation is relatively easy, Hopf local theory was an useful guide for our numerical study of the laser equations.

The multiple branching of solutions does not necessarily lead to a chaotic laser. We have shown that different forms of bistable responses are possible near the first instability of the laser. These cases are of practical interest for semiconductor laser experiments which are mainly based on the determination of power spectra. Jumps between steady, periodic or quasiperiodic regimes can be observed by the changes of frequencies.

We have shown that a secondary bifurcation to stable quasiperiodic oscillations results from the interaction of two successive primary Hopf bifurcations. The new frequency equals the difference between the two individual

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Hopf frequencies. Our analysis encourages investigating the possible secondary bifurcations in LK equations. For LK problem, it is suspected that a secondary bifurcation to unstable quasiperiodic oscillations may explain the coexistence of two stable primary Hopf bifurcation branches [?] [?].

7   Acknowledgments

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8 Appendix A. The small delay limit $(\Theta \to 0)$.

In this appendix, we examine the behavior of the Hopf bifurcation and limit points for small Θ using Eqs. (??) and (??). As $\Theta \to 0$, all Hopf bifurcation branches moves to higher values of B $(B = O(\Theta^{-1}))$ except one which converges to the vertical branch (??). More precisely, we find that

$$R \simeq \sqrt{\frac{6}{\Theta}\left(\frac{C - C_H}{C_H^0 \cot(\theta)}\right)} \text{ and } B \simeq B_0 - \frac{1}{6}R^2 \qquad (33)$$

where the new bifurcation point $C_H \simeq C_H^0$ is shifted as

$$C_H(\Theta) \equiv C_H^0 \left(1 - \Theta B_0 \cot(\theta)\right). \qquad (34)$$

Thus the bifurcation becomes vertical as $\Theta \to 0$ $(C - C_H^0 = O(\Theta)$ if $R = O(1))$. All the other Hopf bifurcation branches moves to infinity as $\Theta \to 0$. We find

$$C_H \simeq \frac{1}{\Theta}\left((1 + 2m)\pi - \theta\right) \qquad (35)$$

$$R = \sqrt{6\left(C - C_H(m)\right)}. \qquad (36)$$

9 Appendix B. Double Hopf bifurcation points.

For critical values of the delay θ (points P_1 and P_4 in Figure 4), a Hopf bifurcation point may exhibits two periodic modes with the distinct frequencies $B = B_1$ and $B = B_2$. From Eq. (??), we find the simple condition $\sin(\theta + B_1\Theta) = \sin(\theta + B_2\Theta)$ which implies that either

$$(B_2 - B_1)\Theta = 2m\pi \qquad (37)$$

where $m = 1, 2, ..$ or

$$2\theta + (B_2 + B_1)\Theta = \pi + 4n\pi \qquad (38)$$

where $n = 0, 1, 2, ...$ We first examine Case (??). From Eq. (??), we know that B_1 and B_2 verify the equations

$$2B_1 + \tan\left((\theta + B_1\Theta)/2\right) = 0 \qquad (39)$$

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$$2B_2 + \tan\left((\theta + B_2\Theta)/2\right) = 0. \tag{40}$$

Substracting (??) and (??), we obtain

$$2(B_1 - B_2) + \frac{\sin\left((B_1 - B_2)\Theta/2\right)}{\cos\left((\theta + B_1\Theta)/2\right)\cos\left((\theta + B_2\Theta)/2\right)} = 0. \tag{41}$$

Using now (??), we find that Eq. (??) cannot be satisfied. We next consider Case (??). We may rewrite the denominator in (??) as a sum of two cosine functions and, using (??), we obtain

$$(B_1 - B_2) + \tan\left((B_1 - B_2)\Theta/2\right) = 0. \tag{42}$$

Eq. (??) is an equation for the difference between the two frequencies. Similarly, we obtain an equation for the sum of the two frequencies by adding (??) and (??). Using (??) again, we find

$$\frac{\pi + 4n\pi - 2\theta}{\Theta} + \frac{1}{\cos\left((B_1 - B_2)\Theta/2\right)} = 0. \tag{43}$$

Introducing $z \equiv (B_1 - B_2)\Theta/2$ and eliminating θ and $\Theta = \beta\theta$ using (??) we obtain an equation for z only:

$$\beta\left[(\pi + 4n\pi)\sin(z) - 2z\right] + 4z\cos(z) = 0. \tag{44}$$

Having z from Eq. (??), we determine θ or $\Theta = \theta/\xi$ by using (??):

$$\Theta = \frac{-2z}{\tan(z)}. \tag{45}$$

For example, with $P = 0.5$ and $T = 1000$, we find $z \simeq 2.28$, $\theta \simeq 61.4$ (equivalently, $\Theta \simeq 3.88$) which gives $B_1 - B_2 \simeq 1.17$. This point corresponds to point P_1 in Figure 4.

We next consider conditions (??) and (??) for the secondary bifurcation point and show that the frequency of the quasiperiodic oscillations is $\mu = (B_1 - B_2)$ at the double Hopf bifurcation point. To this end, we consider Eqs. (??) and (??) evaluated at this point ($R = 0$ and $C = C_H$) :

$$\mu^2 + C_H^2 \sin^2(\mu\Theta/2)\cos(\mu\Theta) + \sin(\mu\Theta)\mu = 0 \tag{46}$$

and

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$$\sin^2(\mu\Theta/2)\left(C_H^2\sin(\mu\Theta)+2\mu\right)=0 \tag{47}$$

Eq. (??) is satisfied if $\sin(\mu\Theta/2)=0$ but from (??), we then find $\mu=0$ which is not possible. The second possibility for satisfying Eq. (??) is to have

$$C_H^2\sin(\mu\Theta)+2\mu=0. \tag{48}$$

With (??), we eliminate C_H in (??) and obtain, after many simplifications, and for $\mu\neq 0$, the following equation for μ

$$\mu+\frac{\sin(\mu\Theta/2)}{\cos(\mu\Theta/2)}=0 \tag{49}$$

which we recognize as Eq. (??) for B_1-B_2. We conclude that $\mu=B_1-B_2$. The solution of the linearized problem is then of the form $A\sim(R_1+c\exp(i\mu\nu))\exp\left[i\left(B_1\nu+d\exp(i\mu\nu)\right)\right]$which we may rewrite as

$$A\sim R_1\exp(iB_1\nu)+(c+idR_1)\exp(iB_2\nu) \tag{50}$$

for small c and d. Thus the secondary bifurcating solution is close to a linear combination of the two periodic modes at the double Hopf bifurcation point.

10 Figure captions

Figure 1. Semiconductor laser subject to an opto-electronic feedback. The figure illustrates the opto-electronic device used by Saboureau et al [?]. The feedback operates on the pump of the laser by using part of the output light which is injected into a photodetector connected to the pump. The delay of the feedback is controlled by changing the length of the optical path.

Figure 2. Bifurcation diagram of the periodic solutions. Figure 2a and Figure 2b represent the amplitude R and the frequency B as functions of the feedback rate C, respectively. Squares denote Hopf bifurcation points. The values of the parameters are $P=0.5$, $T=1000$ and $\tau=2000$ (implying $\theta\simeq 63.25$ and $\Theta=4$).

Figure 3. Hopf bifurcation points. Figure 3a and Figure 3b represent the Hopf bifurcation point C_H and the Hopf frequency B_H as functions of the delay θ, respectively. The values of the fixed parameters are $P=0.5$ and

$T = 1000$. The squares denoted the Hopf bifurcation points for $\tau = 2000$ ($\theta = 63.25$) as shown in Figure 2.

Figure 4. Primary and secondary Hopf bifurcation lines. The Hopf bifurcation lines $C = C_H$ (full lines) and the quasiperiodic bifurcation line $C = C_{QP}$ (dashed line) are represented as functions of the scaled delay θ. The points P_1 and P_4 are double Hopf bifurcation points. The point P_2 corresponds to a change of direction of the Hopf bifurcation point (supercritical for $\theta < \theta_{P_2}$ and subcritical for $\theta > \theta_{P_2}$). At the point P_3, a Hopf bifurcation point and a quasiperiodic bifurcation point may have the same value of the bifurcation parameter C but they exhibit different amplitudes. The values of the fixed parameters are $P = 0.5$, $T = 1000$.

Figure 5. Qualitative bifurcation diagrams. We represents the amplitude of the rapid oscillations (R) as a function of the feedback strength (C). The horizontal line represents the laser steady state. It changes stability at a Hopf bifurcation point which leads to either supercritical oscillations (Figure 5a) or subcritical oscillations (Figure 5b and 5c). In all figures, quasiperiodic oscillations appear as the result of a secondary bifurcation.

Figure 6. Numerical bifurcation diagrams. Figure 6a and 6b represent the bifurcation diagrams of the periodic and quasiperiodic oscillations obtained from the laser equations (??) and (??) and from the amplitude equations (??) and (??), respectively. From (??), we note that $y \simeq 2\epsilon R \sin(s + \phi)$ and we represent $\max(y) = 2\epsilon R$ in Figure 6b. The values of the parameters are the same as in Figure 2. The two main branches of periodic solutions and their successive bifurcations have been obtained by changing η by successive steps first forward and then backward. The dotted lines in Figure 6b correspond to the exact expressions of the branches of periodic solutions given by (??) and (??).